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Perfect Fluid in a Static Isotropic Universe

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Solutions of the Einstein field equations for a perfect fluid in a static isotropic gravitational field are obtained. The solution given by Melnick and Tavensky is corrected.

INTRODUCTION

The purpose of the present work is to improve upon the solutions of the Einstein equations obtained by Melnick and Tavensky (1975) for a perfect fluid. A complete set of exact solutions is obtained in this paper.

1. PROBLEM OF MELNICK AND TAVENSKY

The paper by Melnick and Tavensky (1975) sets out to give a method of obtaining solutions of the Einstein equations for a perfect fluid in comoving coordinates where the energy-momentum tensor is given by

$$T^{\mu\nu} = (p+\omega)u^{\mu}u^{\nu} - pg^{\mu\nu}$$

$$T_1^1 = T_2^2 = T_3^3 = -p, \qquad T_0^0 = \omega \tag{1.1}$$

$$T_i^j = 0 \qquad \text{for} \quad i \neq j$$

p is the pressure, ω is the rest energy density, and the metric considered in this paper will be static and isotropic, that is,

$$ds^2 = \exp[2\phi(x, y, z)]\, dt^2 - \exp[2\psi(x, y, z)]\,(dx^2 + dy^2 + dz^2) \tag{1.2}$$

where $\phi(x, y, z)$ and $\psi(x, y, z)$ are arbitrary functions.

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Melnick and Tavensky (1975) show that for this case the Einstein equations reduce to

$$\Lambda_{,ik} = \Lambda\lambda_{,i}\lambda_{,k} - \Sigma\delta_{ik} \tag{1.3}$$

$$\begin{aligned} -8p\pi &= \tfrac{1}{3}\exp(-2\psi)[2\nabla^2\psi + (\nabla\psi)^2 - 2(\nabla\phi)^2] \\ 8\pi\omega &= \exp(-2\psi)[2\nabla^2\psi + (\nabla\psi)^2] \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} \lambda &= \phi\sqrt{2} \\ \Lambda &= \exp[-(\phi+\psi)] \\ \Sigma &= \tfrac{1}{3}[\nabla^2\Lambda - 2\Lambda(\nabla\phi)^2] \\ (x, y, z) &= (x^1, x^2, x^3) \\ i, k &= 1, 2, 3 \\ \Gamma_{,i} &= \partial\Gamma/\partial x_i \end{aligned} \tag{1.5}$$

The authors then set out to simplify equations (1.3). But in our opinion the work of Melnick and Tavensky (1975) needs significant additions and alterations on several grounds.

First, in order to solve equations (1.3), Melnick and Tavensky (1975) rewrite equations (1.3) as

$$d\Lambda_{,i} = \Lambda\lambda_{,i}\, d\lambda - \Sigma\, dx_i \tag{1.6}$$

From equations (1.6), Melnick and Tavensky (1975) obtain

$$\begin{aligned} \Lambda_{,x} &= F(\lambda, x) = -\Sigma \\ \Lambda_{,y} &= G(\lambda, y) = -\Sigma \\ \Lambda_{,z} &= H(\lambda, z) = -\Sigma \end{aligned}$$

But it will be shown later that the correct result is

$$\begin{aligned} \Lambda_{,x} &= F(\lambda, x) = -x\Sigma + u(\lambda) \\ \Lambda_{,y} &= G(\lambda, y) = -y\Sigma + v(\lambda) \\ \Lambda_{,z} &= H(\lambda, z) = -z\Sigma + w(\lambda) \end{aligned}$$

Obviously this means that much of the rest has to be changed.

Again, the case of $\Sigma = \text{const}$ is not worked out by Melnick and Tavensky (1975), except for an observation that if $\Sigma = \text{const}$, then the $\lambda = \text{const}$ surfaces are planes. (This omission is mentioned by them.)

Instead of making these additions and alterations, it would be simpler to solve equation (1.3) from the start. We shall go further than Melnick and Tavensky (1975) and obtain the complete set of solutions for the case under consideration.

2. SOLUTIONS OF THE PROBLEM

2.1. Special Case $\phi = \text{const}$, i.e., $\lambda = \text{const}$

From equation (1.3), we get

$$\Lambda_{,xy} = \Lambda_{,yz} = \Lambda_{,zx} = 0$$
$$\Lambda_{,xx} = \Lambda_{,yy} = \Lambda_{,zz} = -\Sigma \tag{2.1}$$

From equations (2.1) one can get

$$\Lambda = P(x) + Q(y) + R(z) \tag{2.2}$$

Using (2.2) and equations (2.1), we get

$$\frac{d^2P}{dx^2} = \frac{d^2Q}{dy^2} = \frac{d^2R}{dz^2} = -\Sigma \tag{2.3}$$

which implies $\Sigma = \text{const}$.

Integrating (2.3) and using (2.2), one can solve equations (1.3) for $\lambda = \text{const}$ as

$$\Lambda = -\tfrac{1}{2}(x^2+y^2+z^2)\Sigma + C_1x + C_2y + C_3z + C \tag{2.4}$$

where the C's and Σ are constants.

Also ϕ and ψ are determined from (1.5).

2.2. General Case

We shall prove that the complete solution is given by

$$\Lambda = -\frac{x^2+y^2+z^2}{2}(a_0f + b_0g) + x(a_1f + b_1g)$$
$$+ y(a_2f + b_2g) + z(a_3f + b_3g) - (a_4f + b_4g) \tag{2.5a}$$

$$\lambda = \int \left[\frac{df/d\eta}{\eta \cdot f - g}\right]^{1/2} d\eta \tag{2.5b}$$

where $f(\eta)$ is any arbitrary function of

$$\eta = \frac{a_1x + a_2y + a_3z - a_4 - \frac{1}{2}a_0(x^2+y^2+z^2)}{-b_1x - b_2y - b_3z + b_4 + \frac{1}{2}b_0(x^2+y^2+z^2)} \tag{2.5c}$$

$$g = \int \eta \frac{df}{d\eta} d\eta \tag{2.5d}$$

where a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, and b_4 are all constants such that η is not a constant.

To prove these results, we first establish the following lemmas.

Lemma 1. For $\lambda \neq$ const, there exist Σ, u, v, w, and k which are functions of λ, at least two of which are not linearly related, such that equations (1.3) are equivalent to the following three equations:

$$(x^2+y^2+z^2)\frac{\Sigma_\lambda}{2} - xu_\lambda - yv_\lambda - zw_\lambda + k_\lambda = 0 \tag{2.6a}$$

$$\Lambda = \frac{-(x^2+y^2+z^2)\Sigma}{2} + xu + yv + zw - k \tag{2.6b}$$

$$\frac{\Sigma+\Sigma_{\lambda\lambda}}{\Sigma_\lambda} = \frac{u+u_{\lambda\lambda}}{u_\lambda} = \frac{v+v_{\lambda\lambda}}{v_\lambda} = \frac{w+w_{\lambda\lambda}}{w_\lambda} = \frac{k+k_{\lambda\lambda}}{k_\lambda} \tag{2.6c}$$

Proof. From equations (1.3) we get

$$\Lambda_{xy} = \Lambda\lambda_x\lambda_y \tag{2.7a}$$

$$\Lambda_{xx} = -\Sigma + \Lambda\lambda_x^2 \tag{2.7b}$$

We have similar equations for Λ_{yz}, Λ_{zx}, and so on.

From the above,

$$\frac{\Lambda_{xy}}{\Lambda_{xz}} = \frac{\lambda_y}{\lambda_z}$$

i.e., the partial derivatives of Λ_x with respect to y and z (treating x constant) are proportional to the derivatives of λ with respect to y and z (treating x constant). Hence, if x is treated as a constant, Λ_x is a function of λ, i.e.,

$$\Lambda_x = F(\lambda_{,x}) \tag{2.8a}$$

$$\frac{\partial F(\lambda, x)}{\partial\lambda} = \Lambda\lambda_x \tag{2.8b}$$

Similarly, $\Lambda_y = G(\lambda_{,y})$, $\Lambda_z = H(\lambda_{,z})$, and

$$\frac{\partial G(\lambda, y)}{\partial\lambda} = \Lambda\lambda_y; \qquad \frac{\partial H(\lambda, z)}{\partial\lambda} = \Lambda\lambda_z \tag{2.9}$$

From (2.8a),

$$\Lambda_{xx} = \frac{\partial F(\lambda, x)}{\partial\lambda}\lambda_x + \frac{\partial F(\lambda, x)}{\partial x}$$

By (2.8b),

$$\Lambda_{xx} = \Lambda\lambda_x^2 + \frac{\partial F(\lambda, x)}{\partial x}$$

Therefore, from (2.7b),

$$\frac{\partial F(\lambda, x)}{\partial x} = -\Sigma \tag{2.10a}$$

Similarly, from (2.9)

$$\frac{\partial G(\lambda, y)}{\partial y} = -\Sigma \tag{2.10b}$$

$$\frac{\partial H(\lambda, z)}{\partial z} = -\Sigma \tag{2.10c}$$

Since from (2.10a), Σ is a function of λ and x only, and from (2.10b), Σ is a function of λ and y only, and from (2.10c), Σ is a function of λ and z only, this implies that Σ is a function of λ only, i.e.,

$$\Sigma = \Sigma(\lambda)$$

Therefore, from equation (2.10), we get

$$\begin{aligned} F(\lambda, x) &= -x\Sigma + u(\lambda) \\ G(\lambda, y) &= -y\Sigma + v(\lambda) \\ H(\lambda, z) &= -z\Sigma + w(\lambda) \end{aligned} \tag{2.11}$$

From (2.8b) and (2.11),

$$\Lambda\lambda_x = -x\Sigma_\lambda + u_\lambda$$

Similarly,

$$\Lambda\lambda_y = -y\Sigma_\lambda + v_\lambda, \qquad \Lambda\lambda_z = -z\Sigma_\lambda + w_\lambda \tag{2.12}$$

i.e.,

$$\frac{\lambda_x}{\lambda_y} = \frac{-x\Sigma_\lambda + u_\lambda}{-y\Sigma_\lambda + v_\lambda} \tag{2.13}$$

But

$$\left.\frac{dy}{dx}\right|_{\lambda \text{ and } z \text{ constants}} = -\frac{\lambda_x}{\lambda_y},$$

Therefore, from equation (2.13)

$$(x^2 + y^2)\frac{\Sigma_\lambda}{2} - xu_\lambda - yv = \text{function of } z \text{ and } \lambda$$

Similarly,

$$(y^2+z^2)\frac{\Sigma_\lambda}{2}-yv_\lambda-zw_\lambda=\text{function of } x \text{ and } \lambda$$

$$(z^2+x^2)\frac{\Sigma_\lambda}{2}-zw_\lambda-xu_\lambda=\text{function of } y \text{ and } \lambda$$

Comparing, we get (2.6a).

By virtue of (2.6a), the partial derivatives of Λ are the same as the partial derivatives of the rhs of (2.6b); therefore, they can almost differ by a constant. That constant term can be absorbed into an unspecified function $k(\lambda)$. Hence we get (2.6b).

Again differentiating (2.6a) with respect to x, we get

$$\lambda_x=\frac{x\Sigma_\lambda-u_\lambda}{-\frac{1}{2}(x^2+y^2+z^2)\Sigma_{\lambda\lambda}+xu_{\lambda\lambda}+yv_{\lambda\lambda}+zw_{\lambda\lambda}-k_{\lambda\lambda}}$$

Therefore, from (2.12),

$$\Lambda=\tfrac{1}{2}(x^2+y^2+z^2)\Sigma_{\lambda\lambda}-xu_{\lambda\lambda}-yv_{\lambda\lambda}-zw_{\lambda\lambda}+k_{\lambda\lambda} \tag{2.14}$$

Now, from (2.14) and (2.6b) we get (2.6c).

Therefore, from equations (1.3) we have obtained all three equations of (2.6). Hence, equations (2.6) are necessary for equations (1.3), which can now be established easily by calculating $\Lambda_{,ij}$ from (2.6b) and using (2.6a) and (2.6c). Hence Lemma 1. ■

Lemma 2. If equation (2.6c) holds, then there exist $f=f(\lambda)$ and $g=g(\lambda)$ such that

$$\frac{f_{\lambda\lambda}+f}{f_\lambda}=\frac{g_{\lambda\lambda}+g}{g_\lambda} \tag{2.15a}$$

and

$$\begin{aligned}\Sigma&=a_0f+b_0g\\u&=a_1f+b_1g\\v&=a_2f+b_2g\\w&=a_3f+b_3g\\k&=a_4f+b_4g\end{aligned} \tag{2.15b}$$

where a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3, a_4, and b_4 are all constants.

Proof. We shall prove this by assuming $\Sigma \neq 0$. The proof is similar if $\Sigma = 0$, but at least one of u, v, w, and k is not zero. The case $\Sigma = u = v = w = k = 0$ is trivial.

Let

$$\frac{u}{\Sigma} = \alpha, \qquad \frac{v}{\Sigma} = \beta, \qquad \frac{w}{\Sigma} = \lambda, \qquad \frac{k}{\Sigma} = \delta \tag{2.16}$$

Putting (2.16) into equation (2.6c), we get

$$\frac{\alpha_{\Sigma\Sigma}}{\alpha_\Sigma} + \frac{2}{\Sigma} - \frac{\Sigma}{\Sigma_\lambda^2} = 0$$

$$\frac{\beta_{\Sigma\Sigma}}{\beta_\Sigma} + \frac{2}{\Sigma} - \frac{\Sigma}{\Sigma_\lambda^2} = 0$$

$$\frac{\lambda_{\Sigma\Sigma}}{\lambda_\Sigma} + \frac{2}{\Sigma} - \frac{\Sigma}{\Sigma_\lambda^2} = 0$$

$$\frac{\delta_{\Sigma\Sigma}}{\delta_\Sigma} + \frac{2}{\Sigma} - \frac{\Sigma}{\Sigma_\lambda^2} = 0$$

which implies

$$\frac{\alpha_{\Sigma\Sigma}}{\alpha_\Sigma} = \frac{\beta_{\Sigma\Sigma}}{\beta_\Sigma} = \frac{\lambda_{\Sigma\Sigma}}{\lambda_\Sigma} = \frac{\delta_{\Sigma\Sigma}}{\delta_\Sigma} \tag{2.17}$$

Integrating equations (2.17) and using equations (2.16), we see that Σ, u, v, w, and k can be written in the form of (2.15b). Hence Lemma 2. ■

Now substituting Σ, u, v, w, and k from (2.15b) into equation (2.6b), we get (2.5a).

Further substituting (2.15) in equation (2.6a), we get

$$\frac{g_\lambda}{f_\lambda} = \frac{a_1 x + a_2 y + a_3 z - a_4 - \frac{1}{2} a_0 (x^2 + y^2 + z^2)}{-b_1 x - b_2 y - b_3 z + b_4 + \frac{1}{2} b_0 (x^2 + y^2 + z^2)} \tag{2.18}$$

$$= \eta \quad \text{(say)}$$

taking $g = g(f)$.

Therefore, from (2.15a),

$$f_\lambda^2 = \frac{f \cdot g_f - g}{g_{ff}}$$

Since $g_f = \eta$, i.e.,

$$f_\lambda^2 = \frac{f \cdot \eta - g}{\eta_f}$$

Therefore,

$$f=[(\eta\cdot f-g)f_\eta]^{1/2} \quad\Rightarrow\quad \lambda=\int\left[\frac{df/d\eta}{\eta\cdot f-g}\right]^{1/2} d\eta$$

where η is determined from (2.18) and

$$g=\int \eta\, df=\int \eta \frac{df}{d\eta}\, d\eta$$

which gives (2.5b) and (2.5d).

3. CONCLUSION

We have obtained the complete set of solutions for the Einstein field equations for an energy-momentum tensor given by equation (1.1) and a metric given by equation (1.2). Here the Einstein equations reduce to equations (1.3) and the complete set of solutions are given for the following cases. The solutions obtained here are improvements of the work of Melnick and Tavensky (1975).

Case I. For $\phi=$ const, the solutions of equations (1.3) are obtained from (2.4); then, from (1.5) we get ψ.

Case II. For $\phi\neq$ const, the solutions of equations (1.3) are obtained from (2.5).

REFERENCE

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